§ 7. Integration

§ 7.1 The Riemannian Integral

In this paragraph we define the integral of step functions as a first step. Then we proceed to extend this notion of integration to more general functions through a limiting procedure using step functions.
Let in the following $-\infty < a < b < \infty$.

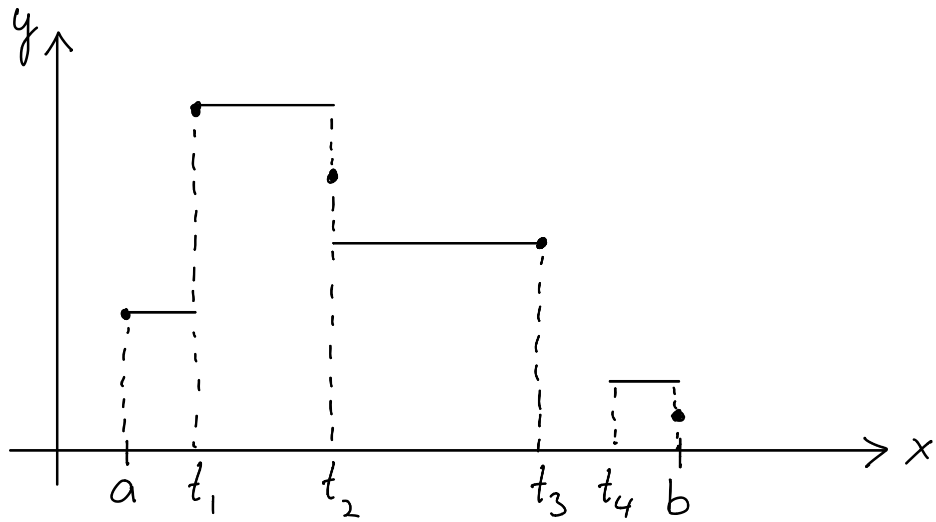
Definition 7.1 (step function):
A function $\varphi : [a,b] \longrightarrow \mathbb{R}$ is called a "step function", if for a decomposition of $I=[a,b]$ into disjoint sub-intervals

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

there are constants $c_1, c_2, \cdots, c_n \in \mathbb{R}$, such that

$$\varphi(x) = c_k \quad \forall\ x \in (t_{k-1}, t_k), \ (1 \leq k \leq n).$$

The values of φ at the points t_k, given by $\varphi(t_k)$, are arbitrary.



Let now $S[a,b]$ be the set of all step functions $\varphi : [a,b] \longrightarrow \mathbb{R}$. We now show that $S[a,b]$ is a sub-vector space of the vector space of all real functions $f : [a,b] \longrightarrow \mathbb{R}$.

<u>Lemma 7.1:</u>

The set $S[a,b]$ is a sub vector space of the vector space of all real functions, i.e.

i) $0 \in S[a,b]$,

ii) $\varphi, \psi \in S[a,b] \Rightarrow \varphi + \psi \in S[a,b]$,

iii) $\varphi \in S[a,b], \lambda \in \mathbb{R} \Rightarrow \lambda\varphi \in S[a,b]$.

Proof:

The properties i) and iii) are trivial. We show here property ii). Let φ be defined through the subdivision

$$Z: a = x_0 < x_1 < \cdots < x_n = b$$

and ψ with respect to the sub-division

$$Z': a = x_0' < x_1' < \cdots < x_m' = b.$$

Now let $a = t_0 < t_1 < \cdots < t_k = b$ be the subdivision of $[a,b]$ which contains all points of Z and Z':

$$\{t_0, t_1, \ldots, t_k\} = \{x_0, x_1, \ldots, x_n\} \cup \{x_0', x_1', \ldots, x_m'\}.$$

Then φ and ψ are constant on every sub-interval (t_{j-1}, t_j), therefore $\varphi + \psi$ is also constant on (t_{j-1}, t_j). Therefore,

$$\varphi + \psi \in S[a,b].$$

$\square$

Definition 7.2 (Integral of step function):

i) Let $\varphi \in S[a,b]$ be defined with respet to the sub-division $a = x_0 < x_1 < \cdots < x_n = b$

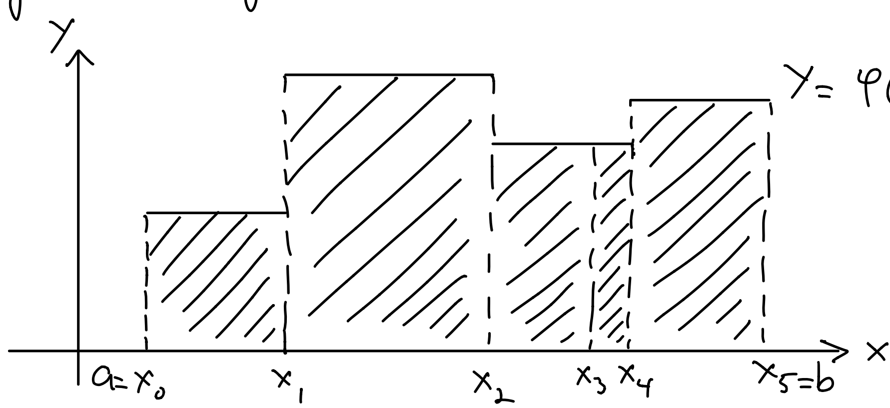
and let $\varphi|_{(x_{k-1},x_k)} = c_k$ for $k=1,\dots,n$.

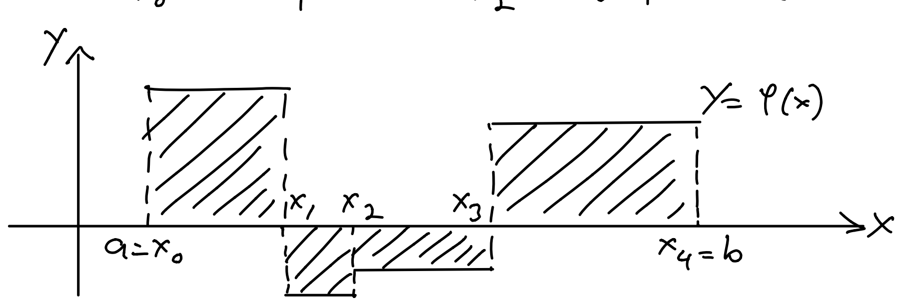
Then set

$$\int_a^b \varphi(x)\,dx := \sum_{k=1}^{n} c_k (x_k - x_{k-1}).$$

Remark 7.1 (Geometric interpretation):

If $\varphi(x) \geq 0$ for all $x \in [a,b]$, one can interpret $\int_a^b \varphi(x)\,dx$ as the area lying between the x-axis and the graph of φ. If φ is negative on some sub-interval, then the corresponding area is counted with a negative sign.





ii) For the integral $\int_a^b \varphi(x)\,dx$ of a step function to be well-defined, one has to show that the definition is independent of the sub-division of the interval $[a,b]$.

Let therefore

$$Z: \quad a = x_0 < x_1 < \cdots < x_n = b,$$

$$Z': \quad a = t_0 < t_1 < \cdots < t_m = b,$$

be two different sub-divisions with

$$\varphi\big|_{(x_{i-1},\, x_i)} = c_i, \qquad \varphi\big|_{(t_{j-1},\, t_j)} = c_j'.$$

Set

$$\int_Z \varphi := \sum_{i=1}^{n} c_i\,(x_i - x_{i-1}), \quad \int_{Z'} \varphi := \sum_{j=1}^{m} c_j'\,(t_j - t_{j-1})$$

We have to show $\int_Z \varphi = \int_{Z'} \varphi$.

<u>Case 1:</u>

Every point in Z is also a point of Z', so $x_i = t_{k_i}$. Then

$$x_{i-1} = t_{k_{i-1}} < t_{k_{i-1}+1} < \cdots < t_{k_i} = x_i, \quad (1 \le i \le n),$$

and $c_j' = c_i$ for $k_{i-1} < j \le k_i$. Thus

$$\int_{Z'} \varphi = \sum_{i=1}^{n} \sum_{j=k_{i-1}+1}^{k_i} c_i (t_j - t_{j-1})$$

$$= \sum_{i=1}^{n} c_i (x_i - x_{i-1}) = \int_{Z} \varphi .$$

Case 2:

Let Z and Z' be arbitrary and let Z^* be the sub-division which contains all points of Z and Z'. Then Case 1 gives:

$$\int_{Z} \varphi = \int_{Z^*} \varphi = \int_{Z'} \varphi .$$

$\square$

Proposition 7.1 (Linearity and Monotomy):

Let $\varphi, \psi \in S[a,b]$ and $\lambda \in \mathbb{R}$. Then we have:

i) $\int_a^b (\varphi + \psi)(x)\,dx = \int_a^b \varphi(x)\,dx + \int_a^b \psi(x)\,dx.$

ii) $\int_a^b (\lambda \varphi)(x)\,dx = \lambda \int_a^b \varphi(x)\,dx.$

iii) $\varphi \leq \psi \Longrightarrow \int_a^b \varphi(x)\,dx \leq \int_a^b \psi(x)\,dx.$

where we have defined for functions $\varphi, \psi : [a,b] \longrightarrow \mathbb{R}$:

$$\varphi \leq \psi \; :\Longleftrightarrow \; \varphi(x) \leq \psi(x) \text{ for all } x \in [a,b].$$

Proof:

According to Remark 7.1 ii), φ and ψ can be defined with respect to the same sub-division of the interval $[a,b]$. The claims of the Proposition are then trivial.

$\square$

Definition 7.2:

Let $f : [a,b] \longrightarrow \mathbb{R}$ be an arbitrary bounded function. Then

$$\overline{\int_a^b} f\,dx := \inf \left\{ \int_a^b \varphi\,dx \;\middle|\; \varphi \in S[a,b],\ \varphi \geq f \right\},$$

$$\underline{\int_a^b} f\,dx := \sup \left\{ \int_a^b \psi(x)\,dx \;\middle|\; \psi \in S[a,b],\ \psi \leq f \right\}$$

are called the "upper", and "lower" "Riemann-Integral" (R-Integral) of f.

Example 7.1:

i) For the step function $\varphi \in S[a,b]$ we have

$$\overline{\int_a^b} \varphi(x)\,dx = \underline{\int_a^b} \varphi(x)\,dx = \int_a^b \varphi(x)\,dx .$$

ii) Let $f:[0,1] \longrightarrow \mathbb{R}$ be the Dirichlet function

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q} . \end{cases}$$

Then we have

$$\overline{\int_0^1} f(x)\,dx = 1 \quad \text{and} \quad \underline{\int_0^1} f(x)\,dx = 0 .$$

Remark 7.2:

In general we have $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx .$

Definition 7.3:

A bounded function $f:[a,b] \longrightarrow \mathbb{R}$ is called "Riemann-integrable" if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx .$$

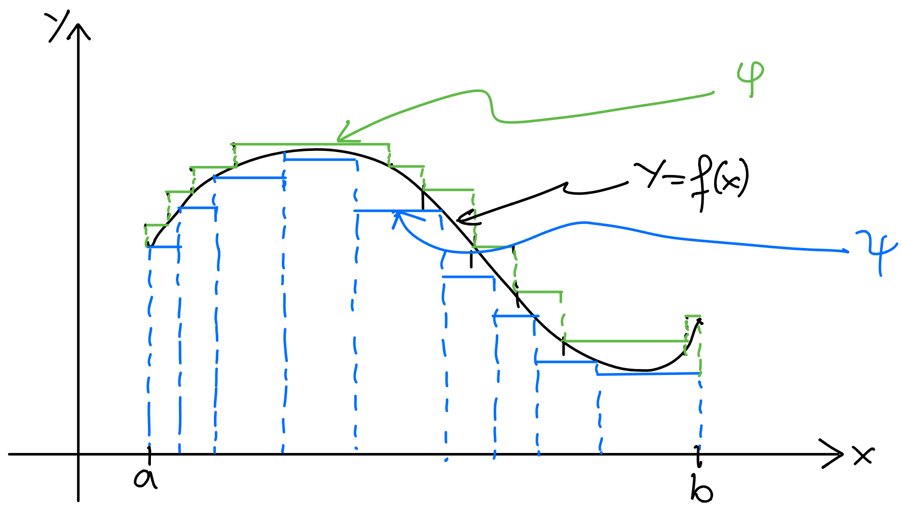
In this case one defines
$$\int_a^b f(x)\,dx := \overline{\int_a^b} f(x)\,dx .$$

Remark 7.3:

This definition coincides with the definition of integral in the case of step functions and is compatible with it.

Indeed, for every step function $\varphi \geq f$ we have $\int_a^b f\,dx \leq \int_a^b \varphi\,dx$, therefore $\int f \leq \overline{\int} f$

Analogously, we get $\int f \geq \underline{\int} f$. Thus, if $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$, it follows $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx = \int_a^b f\,dx$.



Proposition 7.2:

A function $f: [a,b] \longrightarrow \mathbb{R}$ is R-integrable, if and only if for each $\varepsilon > 0$, there exist step functions $\varphi, \psi \in S[a,b]$ s.t.

$$\varphi \leq f \leq \psi$$

and

$$\int_a^b \psi(x)\,dx - \int_a^b \varphi(x)\,dx < \varepsilon.$$

Proof:

This follows from the definition of inf and sup.

$\square$

Proposition 7.3:

Every continuous function $f: [a,b] \longrightarrow \mathbb{R}$ is integrable.

Lemma 7.2:

Let $f: [a,b] \longrightarrow \mathbb{R}$ be continuous. Then for each $\varepsilon > 0$ there exist step functions $\varphi, \psi : [a,b] \longrightarrow \mathbb{R}$ such that:

i) $\psi(x) \leq f(x) \leq \varphi(x)$ for all $x \in [a,b]$.

ii) $|\varphi(x) - \psi(x)| \leq \varepsilon$ for all $x \in [a,b]$.

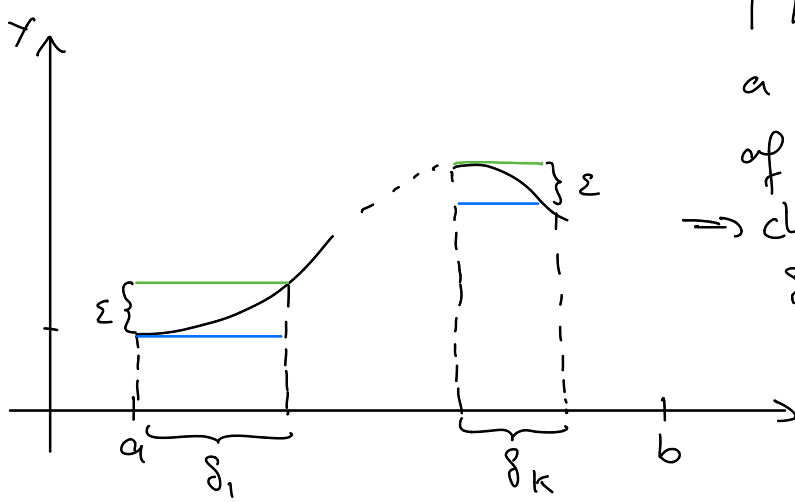
Proof:

As $[a,b]$ is a closed interval, f is bounded. Thus for $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \forall\, x,y \in [a,b] \text{ with } |x-y| < \delta$$

"uniform continuity" (omit the proof here)

in pictures:



The $\{\delta_K\}$ form a finite sub-division of $[a,b]$

$\Rightarrow$ choose

$$\delta := \min_K \delta_K$$

Choose n such that $(b-a)/n < \delta$ and set

$$t_K := a + K\frac{b-a}{n} \quad \text{for } K = 0, \dots, n.$$

This gives a sub-division

$$a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$$

with $t_K - t_{K-1} < \delta$. For $1 \leq K \leq n$ set

$$c_K := \sup\{f(x) \mid t_{K-1} \leq x \leq t_K\},$$

$$c_K' := \inf\{f(x) \mid t_{K-1} \leq x \leq t_K\}.$$

Corollary 4.2 $\Rightarrow$ $c_k = f(\xi_k)$ and $c_k' = f(\xi_k')$ for some points $\xi_k, \xi_k' \in [t_{k-1}, t_k]$ and $|\xi_k - \xi_k'| < \delta$.

Thus
$$|c_k - c_k'| < \varepsilon \quad \forall\, k.$$

We now define step functions $\varphi, \psi : [a,b] \to \mathbb{R}$ as follows:

$$\varphi(x) := c_k, \quad \psi(x) := c_k' \qquad \text{for } t_{k-1} \leq x < t_k, \ (1 \leq k \leq n)$$

$$\varphi(b) := \varphi(t_{n-1}), \quad \psi(b) := \psi(t_{n-1})$$

$\Rightarrow$ With these definitions, conditions i) and ii) are satisfied.

$\square$

Proof Prop. 7.3:

According to Lemma 7.2 there exist for given $\varepsilon > 0$ step functions $\varphi, \psi \in S[a,b]$ with $\psi \leq f \leq \varphi$ and

$$\varphi(x) - \psi(x) \leq \frac{\varepsilon}{b-a} \quad \forall\, x \in [a,b]$$

Thus we get from Prop. 7.1:

$$\int_a^b \varphi(x)\,dx - \int_a^b \psi(x)\,dx = \int_a^b (\psi(x) - \varphi(x))\,dx \leq \int_a^b \frac{\varepsilon}{b-a} = \varepsilon$$

Prop. 7.2 $\Rightarrow$ f is integrable.

$\square$